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Existence of Regular Nilpotent Elements in the Lie Algebra of a Simple Algebraic Group in Bad Characteristics

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Let G denote a connected simple linear algebraic group over the algebraically closed field K. Let $\mathfrak{g}$ denote the Lie algebra of G. In this paper we show that regular nilpotent elements exist in $\mathfrak{g}$ in all characteristics. This answers the question posed by Springer in [4] where he had shown this result to hold only in good characteristics (definition below). We also show that in the adjoint case the centralizers of such elements are always connected. Betty Lou has proved similar results for groups in [2].

Let T be a maximal torus in G. With T there is associated a root system R; the r in R are rational characters of T. For each r in R there exists a morphism X_r from the additive group G_a of K to G which is an isomorphism of G_a onto $X_r(G_a)$ and $X_r(G_a)$ is normalized by T according to the character r. We write X_r for $X_r(G_a)$. X_r is then an unipotent subgroup of G and every element of X_r is of the form $X_r(t_r)$ for some t_r in K. Fixing a set of simple roots, let U be the subgroup of G generated by the X_r with $r>0$. Let B be the group generated by T and U. Then B is a Borel subgroup of G. U is a maximal connected unipotent subgroup of G. The Borel subgroups of G are conjugate; so are the maximal connected unipotent subgroups. The structure of $\mathfrak{g}$ is then as follows: $\mathfrak{g}$ is a direct sum $\mathfrak{g}=\mathfrak{t}+\sum_{r\in R} Ke_r$ where $\mathfrak{t}$ is the Lie algebra of T and Ke_r that of X_r. The Lie algebra of B is $\mathfrak{b}=\mathfrak{t}+\sum_{r>0} Ke_r$ and that of U is $\mathfrak{u}=\sum_{r>0} Ke_r$. G acts on $\mathfrak{g}$ by means of the adjoint representation Ad. For X in $\mathfrak{g}$ we denote by G_X the centralizer of X in G.

Then $\dim G_X \geqslant l$ where $l=\text{rank of } G=\dim T$. [See [4]].

X is said to be *regular* if $\dim G_X=l$.

X is said to be *nilpotent* if X is contained in the Lie algebra of a

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connected unipotent subgroup H of G. Since H is conjugate to a subgroup of U, X is nilpotent if and only if $\mathrm{Ad}(g)(X)$ is in $\mathfrak{u}$ for some g.

Springer has shown in [4] that if char K is either 0 or a good prime for G [definition follows] then regular nilpotent elements exist in $\mathfrak{g}$; this is established by showing that the principal nilpotent element $X=\sum e_r$ (r simple) is regular.

A prime p is said to be a bad prime for G if p divides a coefficient of the highest root in R. The bad primes for the simple root systems of various types are as follows:

$$\begin{aligned} A_l&: \text{ none,} \\ B_l, C_l, D_l&: \ p=2, \\ E_6, E_7, F_4, G_2&: \ p=2, 3, \\ E_8&: \ p=2, 3, 5. \end{aligned}$$

We will calculate the centralizer G_X of the principal nilpotent element X of $\mathfrak{g}$ in the case of bad characteristics for G. Since G_X is the direct product of the centre of G and U_X, the centralizer of X in U [see [4]], it is enough to find U_X. With a fixed ordering of the positive roots of G, any element u in U can be written uniquely as $u=\prod_{r>0} X_r(t_r)$. The height of the root r associated with a variable t_r will be considered frequently, so we define the height of t_r, $\mathrm{ht}(t_r)$, to be the height of r. Now

$$\mathrm{Ad}(u)(X)=\prod_{r>0} \mathrm{Ad}\, X_r(t_r)\, X, \qquad t_r \in K.$$

If r, s are positive roots

$$\mathrm{Ad}\, X_r(t_r)\, e_s = e_s + N_{r,s} t_r e_{r+s} + \frac{N_{r,s} N_{r,r+s}}{2!} t_r^2 e_{2r+s} + \cdots,$$

where the $N_{r,s}$ are the structural constants of the Lie algebra $\mathfrak{g}$, which satisfy the following relations:

(1) $N_{r,s}=0$ if $r+s$ is not a root.

(2) $N_{r,s}=\pm(p+1)$ where p is the smallest positive integer such that $s-pr$ is a root but $s-(p+1)r$ is not a root.

(3) $N_{r,s}=-N_{s,r}$.

(4) $N_{r,s} N_{r+s,t} + N_{s,t} N_{s+t,r} + N_{t,r} N_{t+r,s} = 0$ [Jacobi identity].

Hence setting Ad $u(X)=X$ is equivalent to seeking solutions for a system of polynomials in N variables [N = number of positive roots]. Therefore U_X can be viewed as an algebraic set in K^N. In all cases we can describe the solution as follows:

(1) There are l free variables.

(2 The variables of height one are zero and a nonfree variable of any height can be expressed as a polynomial in variables of lesser heights.

It follows that dim $U_X = l$ and that U_X is connected. Since G_X is the direct product of U_X and the centre of G (see [5]), this will prove the following:

THEOREM. (a) *Regular nilpotent elements exist in bad characteristics and they form a single conjugacy class.*

(b) *If G is adjoint and X is regular nilpotent then G_X is connected.*

(c) *The heights of the free variables mentioned above are independent of the regular nilpotent element chosen and they are given by the table near the end of this paper.*

The conjugacy result is proved as in [4] while the independence in (c) comes from the fact that if $U=U_1 \supset U_2 \supset U_3 \supset \cdots$, denotes the lower central series of U, then each U_i is the product of the root subgroups of height $\geqslant i$, so that the heights of the free variables are just those values of i for which

$$\dim(U_X \cap U_i) \gneqq \dim(U_X \cap U_{i+1}).$$

In this paper we prove the existence in (a) by actual calculations of the centraliser U_X of the principal nilpotent element $X=\sum_{r\,\mathrm{simple}} X_r$ (written earlier as $\sum e_r$) for the different types of groups and their corresponding bad characteristics.

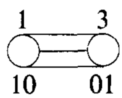
For the classical groups (of type B_n, C_n, and D_n) the results were obtained by direct calculations using the standard representations of these groups and induction on n. For example, we considered C_n to be the group of symplectic transformations on a $2n$-dimensional space relative to the skew-symmetric form whose matrix is

$$I=\begin{bmatrix} & I \\ -I & \end{bmatrix},$$

where I is the $n \times n$ identity matrix. Then $X=\sum_{r\,\mathrm{simple}} X_r$ looks like

$$
\begin{array}{c}
n\times n\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \\
\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}
\end{array}
\left[\begin{array}{cccccc|cccccc}
0 & 0 & 0 & \cdots & & 0 & & & & & & \\
1 & 0 & 0 & & & 0 & & & & & & \\
0 & 1 & 0 & & & 0 & & & & & & \\
\vdots & & & \ddots & & \vdots & & & & & & \\
0 & 0 & 0 & \cdots & 1 & 0 & & & & & & \\
\hline
 & 0 & 0 & \cdots & & 0 & 0 & 1 & 0 & & \cdots & 0 \\
 & 0 & 0 & & & 0 & 0 & 0 & 1 & & & 0 \\
 & \vdots & & \ddots & & \vdots & 0 & 0 & 0 & 1 & & 0 \\
 & 0 & 0 & & & 0 & \vdots & & & & \ddots & \vdots \\
 & 0 & 0 & \cdots & & 1 & 0 & 0 & 0 & & & 1 \\
 & & & & & & 0 & 0 & 0 & & \cdots & 0
\end{array}\right]
\begin{array}{c}
\begin{array}{c} \\ \\ \\ \\ \\ \end{array} \\
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\}n\times n
\end{array}
$$

and its centralizer can be easily calculated. For the exceptional group of type G_2 we label the Dynkin diagram as follows:



We order the positive roots as follows:

	Height
$1-10$	1
$2-01$	1
$3-11$	2
$4-21$	3
$5-31$	4
$6-32$	5

The structure constants $N_{\alpha,\beta}$ are taken as follows:

$$N_{1,2}=1, \qquad N_{3,1}=2, \qquad N_{4,1}=3, \qquad N_{5,2}=1, \qquad N_{4,3}=-3,$$

and $N_{\alpha,\beta}=0$ otherwise.

The principal nilpotent element of the Lie Algebra is $X=X_1+X_2$ where X_1 and X_2 are, respectively, the basis elements corresponding to the simple roots 10 and 01.

Let $u = X_1(A)\ X_2(B)\ X_3(C)\ X_4(D)\ X_5(E)\ X_6(F)$ be any unipotent element in U_X. Then

$$\begin{aligned} u \in U_X &\Leftrightarrow \mathrm{Ad}(u)(X) = X \\ &\Leftrightarrow \mathrm{Ad}(u)(X_1 + X_2) = X_1 + X_2 \\ &\Leftrightarrow \mathrm{Ad}(u)\, X_1 + \mathrm{Ad}(u)\, X_2 = X_1 + X_2. \end{aligned}$$

Let us now calculate

$$\mathrm{Ad}(u)(X_1) = \mathrm{Ad}(X_1(A))\ \mathrm{Ad}(X_2(B)) \cdots \mathrm{Ad}(X_6(F))(X_1),$$

$$\begin{aligned}
&\mathrm{Ad}(X_6(F))(X_1) = X_1, \\
&\mathrm{Ad}(X_5(E))(X_1) = X_1, \\
&\mathrm{Ad}(X_4(D))(X_1) = X_1 + N_{4,1} DX_5 = X_1 + 3DX_5, \\
\mathrm{Ad}(X_3(C))[X_1 + 3DX^5] &= X_1 + N_{3,1} CX_4 + \frac{N_{3,1} N_{3,4}}{2!} D^2 X_6 + 3DX_5 \\
&= X_1 + 2CX_4 + 3DX_5 + 3C^2 X_6, \\
&\mathrm{Ad}(X_2(B))[X_1 + 2CX_4 + 3DX_5 + 3C^2 X_6] \\
&\quad = X_1 + N_{2,1} BX_3 + 2CX_4 + 3DX_5 + 3DN_{2,5} BX_6 + 3C^2 X_6 \\
&\quad = X_1 + BX_3 + 2CX_4 + 3DX_5 + 3(C^2 - BD)\, X_6, \\
&\mathrm{Ad}(X_1(A))[X_1 + BX_3 + 2CX_4 + 3DX_5 + 3(C^2 - BD)\, X_6] \\
&\quad = X_1 + B\left(X_3 + AN_{1,3} X_4 + \frac{A^2}{2!} N_{1,3} N_{1,4} X_5 \right) \\
&\qquad + 2C(X_4 + AN_{1,4} X_5) + 3DX_5 + 3(C^2 - BD)\, X_6 \\
&\quad = X_1 + BX_3 + 2(C - AB)\, X_4 + 3(A^2 B - 2AC + D)\, X_5 \\
&\qquad + 3(C^2 - BD)\, X_6.
\end{aligned}$$

Hence

$$\begin{aligned} \mathrm{Ad}(u)\, X_1 = X_1 + BX_3 + 2(C - AB)\, X_4 + 3(A^2 B - 2AC + D)\, X_5 \\ + 3(C^2 - BD)\, X_6. \end{aligned} \tag{I}$$

Similarly

$$\mathrm{Ad}(u)(X_2) = X_2 - AX_3 + A^2 X_4 - A^3 X_5 + EX_6. \tag{II}$$

Therefore, from (I) and (II) and

$$\mathrm{Ad}(u)(X_1+X_2)=X_1+X_2$$

we get:

(1) coefficient of $X_3 = B - A = 0 \rightarrow B = A$,

(2) coefficient of $X_4 = 2C - 2AB + A^2 = 0$,

(3) coefficient of $X_5 = 3A^2B - 6AC + 3D - A^3 = 0$,

(4) coefficient of $X_6 = 3(C^2 - BD) + E = 0$.

Now the bad primes for a Lie algebra of type G_2 are $p=2$ and $p=3$. For $p=2$ we get from (1) and (2), $A=B=0$; (3) gives $D=0$. Finally (4) yields $E+3C^2=0 \rightarrow E=C^2$. Therefore, the free variables are C and F of height 2 and 5, respectively. The variables A, B of height 1 are 0, the variable of D of height 3 is 0 and the variable E of height 4 is the square of C of height 2.

Hence U_X is connected and X_1+X_2 is regular nilpotent in $\mathfrak{g}$.

For $p=3$ we get from (1) and (2) that $A=B$, $C=-A^2$; (3) gives $A=0$, $A=B=C=0$; (4) gives $E=0$. Thus $A=B=C=E=0$ and the free variables are D and F of heights 3 and 5, respectively.

Hence U_X is connected and X_1+X_2 is regular nilpotent in $\mathfrak{g}$.

The calculations for the groups of type F_4, E_6, E_7, and E_8 were done using computers.

The results obtained for the various types of groups and their corresponding bad characteristics are as follows:

Group type	Bad characteristics	Heights of free variables
G_2	$p=2$	2, 5
	$p=3$	3, 5
F_4	$p=2$	5, 7, 8, 11
	$p=3$	3, 5, 7, 11
E_6	$p=2$	2, 4, 5, 7, 8, 11
	$p=3$	3, 4, 5, 7, 8, 11
E_7	$p=2$	5, 7, 8, 9, 11, 13, 17
	$p=3$	3, 5, 7, 9, 11, 13, 17
E_8	$p=2$	8, 11, 13, 14, 17, 19, 23, 29
	$p=3$	7, 9, 11, 13, 17, 19, 23, 29
	$p=5$	5, 7, 11, 13, 17, 19, 23, 29

In each case the number of free variables is the rank of the group. Hence the element X is regular. Further all other variables were found to be polynomials in free variables of lesser height. Hence the centralizer of X in U is connected. This proves all the assertions in the theorem.

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